Prediction and testing of mixtures of features issued from a continuous dictionary

Clément Hardy
joint work with C. Butucea (CREST, ENSAE, IP Paris), J.-F. Delmas (ENPC), A. Dutfoy (EDF)

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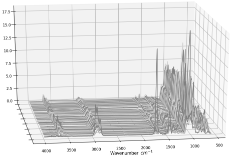
Industrial motivation: spectroscopy

Wave numbers (cm-1)	Peak assignment
3690-3400-3364-3200-3014	-OH
2952-2920-2850	$\nu - CH_2, CH_3$ Aliphatic
1731	$\nu - C = O$
1647	$\nu - C = C$ de $HC = CH_2$
1540	$\nu - C = C$ de R-CR=CH-R, δ CH2 Aliphatic
1419	δCH_2, δ-CH Aliphatic
1160-1082	ν Si-O (SiO_2)
1009-909	ν Si-O (Si-OH)
825	C-Cl
664	CH Aromatic

Location of absorption spikes of chemical components for polychloroprene samples ([Tchalla, 2017]).

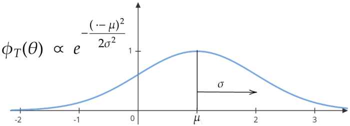
Infrared signals



Continuous dictionary

$$\theta = (\mu, \sigma) \in \Theta = \mathbb{R} \times \mathbb{R}_+^*$$

$$\phi_T(\theta) \propto e^{-\frac{(\cdot - \mu)^2}{2\sigma^2}}$$



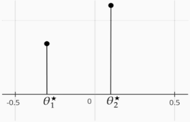
The model

A noisy signal $y = (y(t),\ t \in \mathbb{R})$ is observed on the grid $t_1, \cdots, t_T$:

$$\mathbf{y} = \underbrace{\sum_{k=1}^{s} \beta_k^\star\, \phi_T(\theta_k^\star)}_{\text{mixture of spikes}} + \underbrace{w}_{\text{noise}}.$$

Goal: recover from y the parameters $\beta^\star$ and $\vartheta^\star = (\theta_1^\star, \cdots, \theta_s^\star)$.

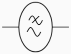
Motivation : low-pass filter



Point sources

$$\mu^\star = \sum_{k=1}^{s} \beta_k^\star \, \delta_{\theta_k^\star}$$

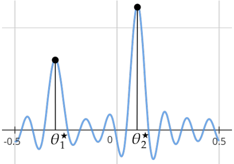
$*$



Low-pass filter (Dirichlet kernel)

$$t \mapsto \sum_{-f_c}^{f_c} \mathrm{e}^{2i\pi k t} = \frac{\sin(\pi\,(2f_c+1)\,t)}{\sin(\pi\,t)}$$

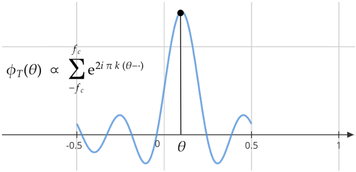
Filtered signal



Continuous dictionary

$\theta \in \Theta = \mathbb{R}/\mathbb{Z}$ and $T = 2f_c + 1$

$$\phi_T(\theta) \propto \sum_{-f_c}^{f_c} \mathrm{e}^{2i\pi k(\theta - \cdot)}$$



The model

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Goal: recover from y the parameters $\beta^\star$ and $\vartheta^\star = (\theta_1^\star, \cdots, \theta_s^\star)$.

Plan of the talk

- **I The model**
- **II Estimation and Prediction**
- **III Simultaneous reconstruction**
- **IV Tests**

I The model

The model

We observe a random element y of a Hilbert space H_T (e.g.: $\mathbb{R}^T$, $L^2(\lambda_T)$,...) with scalar product $\langle \cdot, \cdot \rangle_T$ (and norm $\|\cdot\|_T$).

Model

$$\mathbf{y} = \underbrace{\sum_{k=1}^{s} \beta_k^\star \, \phi_T(\theta_k^\star)}_{\text{signal}} \quad + \quad \underbrace{w_T}_{noise}.$$

Notations

- T increases with the amount of information of the observation (number of observation points, 1/noise level...).
- $\theta_k^\star \in \Theta \subset \mathbb{R}$ and $\beta_k^\star \in \mathbb{R}$, for all k.
- Continuous dictionary $(\phi_T(\theta), \theta \in \Theta)$ of elements of H_T of norm 1. The map ϕ_T is continuous on Θ.
- w_T Gaussian process.

The model: Gaussian noise (I)

Assumptions on the noise (H1)

For all $f \in H_T$, the random variable $\langle f, w_T \rangle_T$ is centered Gaussian with:

$$\mathrm{Var}\left(\langle f, w_T \rangle_T\right) \le \Delta_T \left\| f \right\|_T^2.$$

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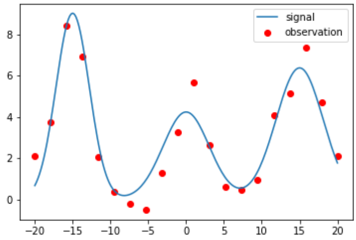
Ex: spectroscopy

- **Regular grid:** $t_1 < \cdots < t_T$ on $\mathbb{R}$ with step-size $\Delta_T = \frac{t_T - t_1}{T}$.
- **Observations:** $y(t_i) = \mathit{signal}(t_i) + w_T(t_i)$, $1 \leq i \leq T$.
- **Noise:** $w_T(t_i)$ i.i.d $\sim \mathcal{N}(0, 1)$.

$H_T = L^2(\lambda_T)$ where $\lambda_T(\mathrm{d}t) = \Delta_T \sum_{j=1}^{T} \delta_{t_j}(\mathrm{d}t)$.



$\Delta_T =$ step-size.

The model: Gaussian noise (II)

Assumptions on the noise (H1)

For all $f \in H_T$, the random variable $\langle f, w_T \rangle_T$ is centered Gaussian with:

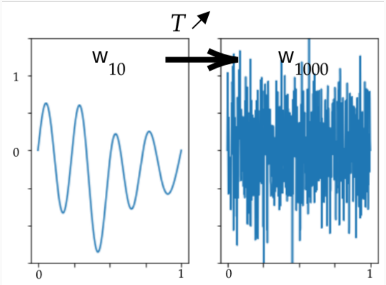
$$\mathrm{Var}\left(\langle f, w_T \rangle_T\right) \leq \Delta_T \, \|f\|_T^2.$$

Ex: low-pass filter

- **Observations:** $(y(t), \, t \in \mathbb{R}/\mathbb{Z})$ s.t $y \in L^2(\mathrm{Leb})$.
- **Truncated white noise:** $w_T = \frac{1}{\sqrt{T}} \sum_{k=1}^{T} G_k \, \psi_k$,

  -G_k i.i.d $\sim \mathcal{N}(0,1)$,
  -$(\psi_k, k \in \mathbb{N})$ o.n.b. of $L^2(\mathrm{Leb})$.

Thus $\|w_T\|_{L^2(\mathrm{Leb})}$ is of order 1 (strong law of large numbers).



$\Delta_T = 1/T$.

II Estimation and Prediction

Estimators

Estimators

$$(\hat{\beta}, \hat{\vartheta}) \in \underset{\beta \in \mathbb{R}^K, \vartheta \in \Theta_T^K}{\operatorname{argmin}} \quad \frac{1}{2} \|y - \beta\, \Phi_T(\vartheta)\|_T^2 + \kappa \|\beta\|_{\ell_1}. \qquad (\mathcal{P}_1(\kappa))$$

- K is an upper bound for s.
- Θ_T is a compact interval.
- $\kappa > 0$ tuning parameter.

$\Phi_T(\vartheta) \in H_T^K$ is defined by:

$$\Phi_T(\vartheta) = (\phi_T(\theta_1), \ldots, \phi_T(\theta_K))^\top.$$

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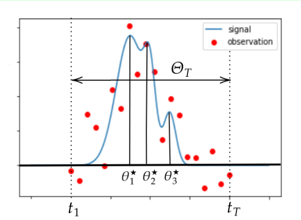
Ex: spectroscopy



Prediction risk

Estimators

Let $(\hat{\beta}, \hat{\vartheta}) \in \mathbb{R}^K \times \Theta^K$ be measurable functions of y solutions of $(\mathcal{P}_1(\kappa))$ "approximating" $(\beta^\star = (\beta_1^\star, \cdots, \beta_s^\star), \vartheta^\star = (\theta_1^\star, \cdots, \theta_s^\star))$. We define:

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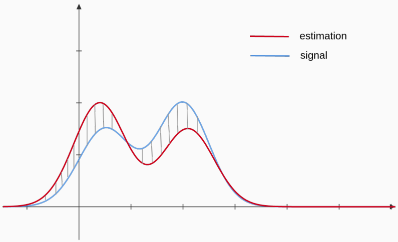
Prediction risk

$$\left\| \beta^\star \Phi_T(\vartheta^\star) - \hat{\beta}\, \Phi_T(\hat{\vartheta}) \right\|_T,$$

where $\Phi_T(\hat{\vartheta}) \in H_T^K$ is defined by:

$$\Phi_T(\hat{\vartheta}) = (\phi_T(\hat{\theta}_1), \ldots, \phi_T(\hat{\theta}_K))^\top,$$



Estimation: estimation risks (I)

Estimators

Let $(\hat{\beta}, \hat{\vartheta}) \in \mathbb{R}^K \times \Theta^K$ be measurable functions of y solutions of $(\mathcal{P}_1(\kappa))$ "approximating" $(\beta^\star = (\beta_1^\star, \cdots, \beta_s^\star), \vartheta^\star = (\theta_1^\star, \cdots, \theta_s^\star))$. We define:

Estimation risks (I)

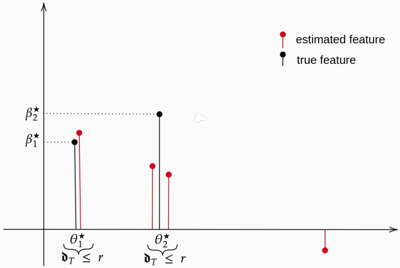
$$\sum_{k=1}^{s} \left| \beta_k^\star - \sum_{\ell \in S_k(r)} \hat{\beta}_\ell \right| \quad \text{and} \quad \sum_{\ell \in S(r)^c} |\hat{\beta}_\ell|,$$

where the set $S(r)$ is given by:

$$S(r) = \bigcup_{1 \le k \le s} S_k(r),$$

with

$S_k(r) = \left\{ \ell,\ \hat{\beta}_\ell \neq 0 \text{ and } \mathfrak{d}_T(\hat{\theta}_\ell, \theta_k^\star) \le r \right\}.$



Estimation: estimation risks (II)

Estimators

Let $(\hat{\beta}, \hat{\vartheta}) \in \mathbb{R}^K \times \Theta^K$ be measurable functions of y solutions of $(\mathcal{P}_1(\kappa))$ "approximating" $(\beta^\star = (\beta_1^\star, \cdots, \beta_s^\star), \vartheta^\star = (\theta_1^\star, \cdots, \theta_s^\star))$. We define:

Estimation risks (II)

$$\sum_{k=1}^{s} \left| |\beta_k^\star| - \sum_{\ell \in S_k(r)} |\hat{\beta}_\ell| \right|,$$

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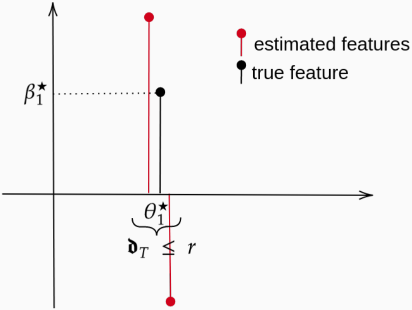

Estimation: the Beurling Lasso (I)

The Beurling Lasso [De Castro & Gamboa, 2012]

$$\min_{\mu \in \mathcal{M}(\Theta_T)} \quad \frac{1}{2}\|y - \langle \phi_T, \mu\rangle\|_T^2 + \kappa\|\mu\|_{TV}. \qquad (\mathcal{P}_2(\kappa))$$

- $\mathcal{M}(\Theta_T)$ the set of measures on Θ_T.
- $\langle \phi_T, \mu\rangle = \int \phi_T(\theta)\, \mu(\mathrm{d}\theta)$.
- $\|\cdot\|_{TV}$ the total variation of a norm.
- $\kappa > 0$ tuning parameter.

Remark

Let $\mu = \sum_{k=1}^K \beta_k\, \delta_{\theta_k}$ (atomic measure), then:

$$\frac{1}{2}\|y - \langle \phi_T, \mu\rangle\|_T^2 + \kappa\|\mu\|_{TV} = \frac{1}{2}\|y - \beta\, \mathbf{\Phi}_T(\vartheta)\|_T^2 + \kappa\|\beta\|_{\ell_1}.$$

Estimation: the Beurling Lasso (II)

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$\mathcal{P}_1(\kappa)$ **and** $\mathcal{P}_2(\kappa)$

- Existence of a solution to $\mathcal{P}_2(\kappa)$ [Bredies & Pikkarainen, 2013].
- If $\mathcal{P}_2(\kappa)$ admits a solution $\hat{\mu} = \sum_{k=1}^{K} \hat{\beta}_k \delta_{\hat{\theta}_k}$ then $(\hat{\beta}, \hat{\vartheta})$ is a solution to $\mathcal{P}_1(\kappa)$.
- If H_T has finite dimension K, then there exists a solution to $\mathcal{P}_2(\kappa)$ with K atoms at most, [Boyer et al, 2019].

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Ex : low-pass filter

The observation space is $H_T = L^2(\mathrm{Leb})$.

Estimation: the Beurling Lasso (II)

The Beurling Lasso [De Castro & Gamboa, 2012]

$$\min_{\mu \,\in\, \mathcal{M}(\Theta_T)} \quad \frac{1}{2}\left\|y - \langle \phi_T, \mu\rangle\right\|_T^2 + \kappa\|\mu\|_{TV}. \qquad (\mathcal{P}_2(\kappa))$$

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Numerical implementation

- Modified Frank-Wolfe algorithm [Boyd, Schiebinger & Recht, 2017], [Denoyelle, Duval, Peyré & Soubies 2020], [Globabae & Poon, 2022].

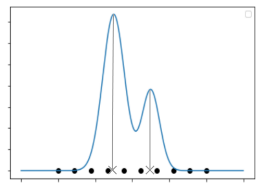
Estimation: why gridless methods

Estimation on a grid on the space of parameters

- Discrete grid of K points $\vartheta^{\mathcal{G}} = (\theta_1^{\mathcal{G}}, \cdots, \theta_K^{\mathcal{G}})$.
- Solve the Lasso problems:

$$\hat{\beta} \in \underset{\beta \in \mathbb{R}^K}{\operatorname{argmin}} \quad \frac{1}{2} \|y - \beta \Phi_{\mathcal{G}}\|_T^2 + \kappa \|\beta\|_{\ell_1},$$

where $\Phi_{\mathcal{G}} = (\phi_T(\theta_1^{\mathcal{G}}), \cdots, \phi_T(\theta_K^{\mathcal{G}}))^\top$.



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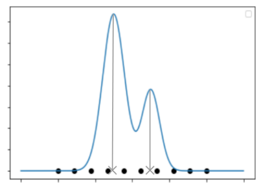


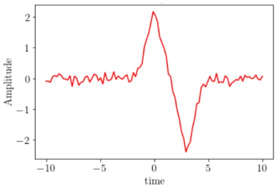
Inconvenients related to the refinement of the grid on Θ

- Strong correlations between the lines $\Phi_{\mathcal{G}} \implies$ numerical problems.
- The size of the grid grows exponentially with d when $\Theta \subset \mathbb{R}^d$.
- In the location model: clusters of spikes in the neighbourhood of the true spikes [Duval & Peyré, 2017].

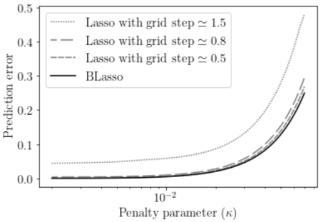
Estimation: numerical aspects (Blasso V Lasso on a grid)

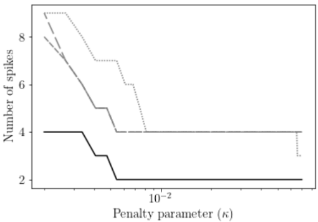
Numerical experiment

- Signal in $H_T = \mathbb{R}^T$ with $T = 100$, mixture of two Gaussian-shaped spikes with $\theta_1^\star = 0$ and $\theta_2^\star = 3$ and amplitudes in [-10,10] uniformly distributed, corrupted by i.i.d. centered Gaussian r. v. with $\sigma = 0.1$.



Results (GitHub: ClementHardy/PySFW)





Estimation: gridless methods

Bibliography

- **BLasso**: [De Castro and Gamboa, 2012], [Bredies & Pikkarainen, 2013].

- **Super-resolution** and **compressed sensing**: [Candès and Fernandez-Granda, 2013, 2014] (BLasso), [Bhaskar, Tang & Recht, 2013] (*atomic norm denoising*)...

- **Prediction and estimation** (dictionary composed of complex-valued Fourier basis): [Tang, Bhaskar & Recht 2014], [Boyer, De Castro & Salmon 2017].

- **Recover the support of a mesure and robustness of BLasso**: [Duval & Peyré, 2015] (spike deconvolution with weak noise $\|w_T\|_T \ll 1$) .

- **General geometric framework for BLasso**: [Poon, Keriven & Peyré, 2021].

- **Mixture (density) model**: [De Castro, Gadat, Marteau & Maugis, 2020].

Estimation: assumptions (I)

Smoothness of the dictionary functions (H2)

- $\varphi_T : \Theta \to H_T$ is $\mathcal{C}^3$.
- $\|\varphi_T(\theta)\|_T > 0$ on Θ.
- $\phi_T(\theta) = \varphi_T(\theta)/\|\varphi_T(\theta)\|_T$.
- $\|\partial_\theta \phi_T(\theta)\|_T^2 > 0$ on Θ.

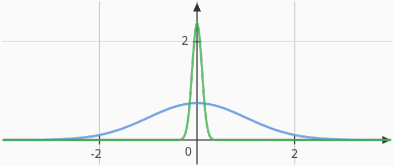
Ex: spectroscopy

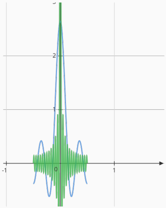
$$\varphi_T(\theta) = \mathrm{e}^{-(\theta-\cdot)^2/2\sigma_T^2}, \quad \theta \in \Theta = \mathbb{R}.$$

Ex: low-pass filter

$$\varphi_T(\theta) = \frac{\sin(\pi(\theta-\cdot)/\sigma_T)}{\sin(\pi(\theta-\cdot))},$$

$$\theta \in \Theta = \mathbb{R}/\mathbb{Z},\ \sigma_T = \frac{1}{T} = \frac{1}{2f_c+1}, \quad T \in 2\mathbb{N}^* + 1.$$





Estimation: assumptions (II)

Kernel and approximating kernel

We define a kernel on Θ^2 to measure the correlation between components of the dictionary:

$$\mathcal{K}_T(\theta, \theta') = \langle \phi_T(\theta), \phi_T(\theta') \rangle_T,$$

and an approximating symmetric kernel $\mathcal{K}^{\text{prox}}$ sur Θ^2_∞.

Estimation: assumptions (II)

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Ex: spectroscopy

If $\Delta_T \to 0$ and $\sigma_T = cst$, $\qquad \mathcal{K}^{\text{prox}} = \lim_{T \to +\infty} \mathcal{K}_T$.

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Assumptions on the approximating (H3)

The kernel $\mathcal{K}^{\text{prox}}$ is $\mathcal{C}^{3,3}$ with bounded derivatives + other **smoothness** assumptions. It is **locally concave on the diagonal** and strictly less than 1 outside the diagonal.

Estimation: assumptions (III)

Fisher-Rao metric on the space of parameters

$$\mathfrak{d}_{\mathcal{K}}(\theta, \theta') = \inf_{\gamma} \int_0^1 |\dot{\gamma}_s| \sqrt{\partial_{x,y}\mathcal{K}(\gamma_s, \gamma_s)} \,\mathrm{d}s$$

inf. on the set of smooth paths $\gamma : [0,1] \to \Theta$ such that $\gamma_0 = \theta$ and $\gamma_1 = \theta'$.

$\to$ **invariance** $\mathfrak{d}_{\mathcal{K}_\varphi}(\theta, \theta') = \mathfrak{d}_{\mathcal{K}_{\varphi \circ h}}(h^{-1}(\theta), h^{-1}(\theta'))$.

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Univariate examples

- **Translated spikes model (spectroscopy / low-pass filter):**

$$\mathfrak{d}_{\mathcal{K}_T}(\theta,\theta') \sim |\theta-\theta'| \text{ (Euclidean distance).}$$

- **Scale model:** $H = L^2(\mathrm{Leb})$ and $\varphi(\theta) = e^{-\cdot\theta}$ with $\Theta = \mathbb{R}_+^*$ and

$$\mathfrak{d}_{\mathcal{K}_T}(\theta,\theta') \propto |\log(\theta/\theta')|.$$

  We have $\mathfrak{d}_{\mathcal{K}_T}(\theta,\theta+\varepsilon) \underset{\theta\to 0}{\to} +\infty$ ($\neq$ Euclidean distance).

Estimation: assumptions (IV)

Proximity of $\mathcal{K}_T$ and $\mathcal{K}^{\text{prox}}$

- **Proximity between kernels:**

$$\mathcal{V}_T = \max_{i,j\in\{0,\cdots,3\}} \sup_{\Theta_T^2} |\mathcal{K}_T^{[i,j]} - \mathcal{K}^{\text{prox}[i,j]}|.$$

- **equivalent metrics** : $\mathfrak{d}_{\mathcal{K}_T}$ and $\mathfrak{d}_{\mathcal{K}^{\text{prox}}}$: $\quad \mathfrak{d}_{\mathcal{K}^{\text{prox}}} / \rho_T \leq \mathfrak{d}_{\mathcal{K}_T} \leq \rho_T\, \mathfrak{d}_{\mathcal{K}^{\text{prox}}}$

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Proximity assumptions between $\mathcal{K}_T$ and $\mathcal{K}^{\text{prox}}$ (H4)

$$s\, \mathcal{V}_T \leq C \quad \text{and} \quad \rho_T \leq \rho.$$

$\rightarrow$ This often rewrites : $T \geq sT_0$!

Estimation: bounds on prediction and estimation errors

Theorem 1

We observe $y \in H_T$ with unknown parameters $\beta^\star \in \mathbb{R}^s$ and $\vartheta^\star = (\theta_1^\star, \cdots, \theta_s^\star) \in \Theta_T^s$ avec $s \leq K$ such that assumptions H1-H4 are checked and, for all $\ell \neq k$,

$$\mathfrak{d}_{\mathcal{K}_T}(\theta_\ell^\star, \theta_k^\star) \gtrsim \delta(s) \ \textbf{(séparation)}.$$

Then, the estimators $\hat{\beta}$ and $\hat{\vartheta}$ defined by $\mathcal{P}_1(\kappa)$ with

$$\kappa \geq \mathcal{C}_1 \sqrt{\Delta_T \log \tau} \quad \text{and} \quad \tau > 1,$$

with **the following bounds on the prediction and estimation risks:**

$$\left\| \beta^\star \Phi_T(\vartheta^\star) - \hat{\beta}\, \Phi_T(\hat{\vartheta}) \right\|_T \leq \mathcal{C}_0 \sqrt{s}\, \kappa,$$

$$\sum_{k=1}^{s} \left| |\beta_k^\star| - \sum_{\ell \in S_k(r)} |\hat{\beta}_\ell| \right| \; + \; \sum_{k=1}^{s} \left| \beta_k^\star - \sum_{\ell \in S_k(r)} \hat{\beta}_\ell \right| \; + \; \left\| \hat{\beta}_{S(r)^c} \right\|_{\ell_1} \leq \mathcal{C}_0\, \kappa\, s.$$

with probability larger than: $1 - \mathcal{C}_2 \left(\frac{|\Theta_T|_{\mathfrak{d}_T}}{\tau \sqrt{\log \tau}} \vee \frac{1}{\tau} \right)$.

Remark: the bounds do not depend on K!

Estimation: separation between parameters

Separation condition

$$\mathfrak{d}_{\mathcal{K}_T}(\theta_\ell^\star, \theta_k^\star) \gtrsim \delta(s),\ \ell \neq k.$$

Estimation: separation between parameters

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$$\mathfrak{d}_{\mathcal{K}_T}(\theta^\star_\ell, \theta^\star_k) \gtrsim \delta(s),\ \ell \neq k.$$

Definition of the separation $\delta(s)$

$$\delta(s) = \inf\Big\{\delta > 0 :\ \max_{1\leq \ell \leq s} \sum_{k=1, k\neq \ell}^{s} |\mathcal{K}^{\mathrm{prox}[i,j]}(\theta_\ell, \theta_k)| \leq u \quad \text{for all}$$

$$(i,j) \in \{0,1\}\times\{0,1,2\} \text{ and } (\theta_1, \cdots, \theta_s) \in \Theta^s \quad \text{s.t.} \quad \mathfrak{d}_{\mathcal{K}^{\mathrm{prox}}}(\theta_\ell, \theta_k) > \delta, \ell \neq k\Big\}.$$

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Ex: spectroscopy

$$\mathfrak{d}_{\mathcal{K}_T}(\theta, \theta') \sim \frac{|\theta - \theta'|}{\sigma_T} \quad \text{and} \quad \delta(s) \lesssim 1/u, \text{ for some } u > 0.$$

Estimation: bounds on the prediction risk

Ex: spectroscopy

- **Grid:** $t_1 < \cdots < t_T$ regular on $\mathbb{R}$ with step-size Δ_T.
- **Noise:** $w_T(t_j)$ i.i.d $\sim \mathcal{N}(0,1)$.
- **Dictionary:** $(\varphi_T(\theta) = \mathrm{e}^{-(\theta-\cdot)^2/2\sigma_T^2}, \theta \in \Theta = \mathbb{R})$.
- **Separation:** $|\theta_\ell^\star - \theta_k^\star| \gtrsim \sigma_T$ for $\ell \neq k$.

We have for $\sigma_T \gtrsim \Delta_T$:

$$\frac{1}{\sqrt{T}}\left\|\hat{\beta}\Phi_T(\hat{\vartheta}) - \beta^\star\Phi_T(\vartheta^\star)\right\|_{\ell_2} \lesssim \sqrt{\frac{s\log(T)}{T}},$$

with probability larger than $1 - C\left(\frac{t_T - t_1}{\sigma_T\, T\sqrt{\log(T)}} \vee \frac{1}{T}\right)$.

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$\rightarrow$ **The upper bound is of the same order as that for the Lasso estimator in the linear regression model (i.e when nonlinear parameters are known).**

Estimation: bounds on the prediction risk

Ex: low-pass filter

- **Truncated white noise:** $w_T = \frac{1}{\sqrt{T}} \sum_{k=1}^{T} G_k \, \psi_k$.
- **Dictionary:** $(\varphi_T(\theta) = \frac{\sin(T\pi(\theta - \cdot))}{\sin(\pi(\theta - \cdot))}, \, \theta \in \Theta = \mathbb{R}/\mathbb{Z})$.
- **Separation:** $|\theta_\ell^\star - \theta_k^\star| \gtrsim \delta(s)/T$ pour $\ell \neq k$.

We have for $T \gtrsim s$:

$$\left\| \hat{\beta} \Phi_T(\hat{\vartheta}) - \beta^\star \Phi_T(\vartheta^\star) \right\|_{L^2(\mathrm{Leb})} \lesssim \sqrt{\frac{s \log(T)}{T}},$$

with probability larger than $1 - \frac{C}{T\sqrt{\log(T)}}$.

III Simultaneous reconstruction

Common structure of the signals

Model

We observe n signals $(Y(i) \in H_T,\ 1 \leq i \leq n)$ s.t **the union of their features is a set of cardinal** s:

$$Y(i) = \sum_{k=1}^{s} B_k^\star(i)\, \phi_T(\theta_k^\star) + W_T(i) \quad \text{for} \quad 1 \leq i \leq n,$$

$$W_T(i) \quad \text{i.i.d} \quad \sim w_T.$$

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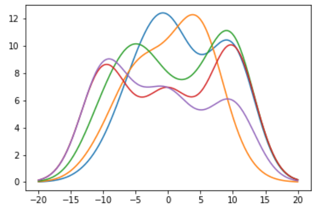
Question

Is it possible to achieve greater efficiency in reconstruction when the signals share most of their features ?



Simultaneous reconstruction

Estimators

$$(\hat{B}, \hat{\vartheta}) \in \underset{B \in \mathbb{R}^{n\times K}, \vartheta \in \Theta_T^K}{\operatorname{argmin}} \quad \frac{1}{2n} \sum_{i=1}^{n} \| Y(i) - B(i) \Phi_T(\vartheta) \|_T^2 + \kappa \sum_{k=1}^{K} \sqrt{\sum_{i=1}^{n} B_k(i)^2}$$

Bibliography

- Group-Lasso for linear models [Yuan & Lin, 2006], minimax bounds [Lounici, Pontil, van de Geer & Tsybakov, 2011].
- Group-BLasso [Golbabae & Poon, 2022].

Simultaneous reconstruction

Theorem

Under H1-H4 and provided $\ell \neq k$,

$$\mathfrak{d}_T(\theta^\star_\ell, \theta^\star_k) \gtrsim \delta(s) \text{ } (\textbf{separation}).$$

Then, for any $\tau > 1$ and a tuning of κ, we get **the bound:**

$$\frac{1}{n} \sum_{i=1}^{n} \left\| B^\star(i)\, \Phi_T(\vartheta^\star) - \hat{B}(i)\, \Phi_T(\hat{\vartheta}) \right\|_T^2 \lesssim s\, \Delta_T \left(1 + \frac{\log(\tau)}{n} \right),$$

with prob. greater than: $1 - \mathcal{C}_2 \left(\frac{1}{\tau^2 \log(\tau)} + \frac{|\Theta_T|_{\mathfrak{d}_T}\, \mathrm{e}^{-n/3}}{\tau \sqrt{\log(\tau)}} \right)$.

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$\rightarrow$ **The upper bound is of the same order as that for the Group-Lasso estimator in the multi-task linear regression model (i.e when nonlinear parameters are known).**

III Tests

Signal detection

Signal detection:

Let $(\beta^\star, \vartheta^\star) \in (\mathbb{R}^*)^s \times \Theta_T^s$ (**unknown**).

$$\begin{cases} H_0 & : \quad s = 0, \\ H_1(\rho) & : \quad \|\beta^\star\|_{\ell_2} \geq \rho. \end{cases}$$

How large does ρ have to be to distinguish these hypotheses?

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Recall

A test Ψ is a measurable function y taking values in $\{0, 1\}$:
$\Psi = 0$ **accept** H_0 and $\Psi = 1$ **reject** H_0.

- **The maximal testing risk**:

$$R_\rho(\Psi) = \underbrace{\sup_{(\beta^\star,\vartheta^\star)\in H_0} \mathbb{P}_{(\beta^\star,\vartheta^\star)}(\Psi = 1)}_{\text{1st type error prob.}} + \underbrace{\sup_{(\beta^\star,\vartheta^\star)\in H_1(\rho)} \mathbb{P}_{(\beta^\star,\vartheta^\star)}(\Psi = 0)}_{\text{2nd type error prob.}}.$$

- **The minimax separation distance for testing at $\alpha \in (0, 1)$**:

$$\rho^\star(\alpha) = \inf\{\rho > 0 : \inf_{\Psi} R_\rho(\Psi) \leq \alpha\}.$$

Signal detection

Framework

- Dictionary: $(\varphi_T(\theta) = h(\theta - \cdot, \sigma_T),\ \theta \in \Theta)$.
- Discrete process on a regular grid of T points on $\mathbb{R}/\mathbb{Z}$ with $w_T(t_j)$ i.i.d $\sim \mathcal{N}(0,1)$ for $1 \leq j \leq T$.

Proposition 1

Under the assumptions H1-H4, assuming $|\theta^\star_\ell - \theta^\star_k| \gtrsim \sigma_T\, \delta(s) \quad \forall \ell \neq k$, we have for $|\Theta_T|/\sigma_T \geq 1$ and $\alpha \in (0,1)$:

$$\rho^\star(\alpha) \lesssim \min\left(\frac{1}{(\alpha T)^{\frac{1}{4}}}, \sqrt{\frac{s}{T}\log\left(\frac{c}{\alpha\, \sigma_T}\right)}\right).$$

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Remarks

- In the sparse linear regression model, a third regime appears with the rate $\frac{p^{1/4}}{\sqrt{T}}$; Ingster, Verzelen, Tsybakov (2010), Nickl and van de Geer, (2013), showed that the minimax testing rate is $\frac{1}{T^{1/4}} \wedge \sqrt{\frac{s\log(p)}{T}} \wedge \frac{p^{1/4}}{\sqrt{T}}$.

Conclusion and perspectives

Estimation and tests

Under least separation conditions between the true non linear parameters:

- Prediction risks of the same order as for the Lasso-type estimators where $\vartheta^\star$ is known.
- Simultaneous reconstruction (analogous to group-Lasso) when many signals share a common structure.
- Testing separation rate is of the same order as for signal detection with $\vartheta^\star$ given.

Perspectives

- $\Theta \subseteq \mathbb{R}^d$
- Improve on the non-linear parameter separation conditions in general
- Extend the testing problems